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Polynomial Invariant of Knots and Links from Two-Parameter Quantum Groups

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A new link polynomial is obtained by introducing a two-parameter Hecke algebra associated with the two-parameter quantum groups.

1. INTRODUCTION

The quantization of the universal enveloping algebra is defined as a one-dimensional deformation of this universal enveloping algebra with one-parameter q. Drinfeld (1986) showed that this requirement means that the deformed universal enveloping algebra is generated by operator-valued matrix functions generating a noncommutative and noncocommutative Hopf algebra which is called the q-parameter quantum group. The representation of the quantum group has a crucial property which relates its structure to the representation theory of Hecke algebra given by the commutant of the deformed universal enveloping algebra, which is given by

$$T_i T_j = T_j T_i \qquad \text{if} \quad |i - j| \geq 2$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$(T_i - q^{-1})(T_i + q) = 0$$

Recently, attention was given to the two-parameter deformation $qAB - pBA = C$, which is in correspondence with the so-called two-parameter quantum group (Georgelin *et al.*, 1996; Cho *et al.*, 1997). If we look for the quantum groups as symmetry groups in the quantum planes, this shows that there can be many quantum planes for a given two-parameter quantum group,

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i.e., $R(p, q)$ is not unique. The algebraic structure of the two-parameter quantum group as a symmetry group of the two-parameter quantum plane has been constructed (Georgelin *et al.*, 1996; Choe *et al.*, 1997). The aim of this paper is to construct the corresponding Hecke algebra, then to introduce a corresponding knot polynomial of 3-variables.

2. TERMINOLOGY

A braid on n strings is an embedding of n oriented intervals (strings) in $D^2 \times I$ joining n points in the disc $D^2 \times \{0\}$ to n points in $D^2 \times \{1\}$. The strings may be permuted, but the last coordinate increases monotonically on each string (Fig. 1a). The set of all such braids form a group, the Artin braid group B_n, which is represented by generators σ_i (Fig. 1b), $i = 1, 2, \ldots, n - 1$, and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i - j| \geq 2 \tag{1}$$

$$\sigma_i\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\sigma_{i+1} \tag{2}$$

The closure $\hat{\alpha}$ of a braid α is formed by joining the top points to the bottom as shown in Fig. 1c, and the braid $\alpha \in B_n$ is called the n-braid representative of the knot (or link) $L \approx \hat{\alpha}$. For more details about braids and links we refer to Birman (1974).

The translation of the equivalence relation of the isotopy of links into an algebraic problem about braids was done by Markov (see Birman, 1974), who showed that the closure of braids which are equivalent under a sequence of moves of the following types are isotopic links:

$$(\alpha, n) \to (\beta\alpha\beta^{-1}, n) \qquad \text{for any} \quad \beta \in B_n \tag{3}$$

$$(\alpha, n) \to (\alpha\sigma_n^{\pm 1}, n + 1) \tag{4}$$

Two knots K_1 and K_2 in $S^3(R^3)$ belong to the same isotopy type if there exists an orientation-preserving homeomorphism of S^3 which maps K_1 onto K_2; for isotopic knots we denote $K_1 \approx K_2$. Jones (1985) discovered a polynomial invariant of the isotopy type of an oriented knot or link; the Jones polynomial was generalized to the two-variable polynomial invariant (Freyd

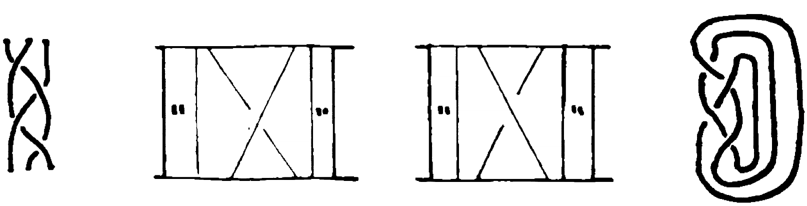

Fig. 1. (a) Geometric braid; (b) positive and negative crossings; (c) closed braid.

et al., 1985). In this paper we introduce a polynomial invariant of the isotopy type of oriented knots and links, by introducing a two-parameter Hecke algebra associated with the two-parameter quantum groups.

Two-parameter quantum groups were constructed in Georgelin *et al.* (1996) and Cho *et al.*, 1997). The corresponding R-matrix is

$$R = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & q - p^{-1} & qp^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}$$

Then

$$R^{-1} = \begin{bmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & pq^{-1} & -pq^{-1}(q - p^{-1}) & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix}$$

3. THE VARIABLE POLYNOMIAL VIA THE HECKE ALGEBRA FROM THE TWO-PARAMETER QUANTUM GROUPS.

Now consider $g_i = P^{-1}R_i^{-1}$, where $R_i = I \otimes I \otimes \ldots \otimes R \otimes I \otimes \ldots \otimes I$ and R is the above matrix in the ith position. Then the reduction relation takes the form

$$(g_i + 1)\left(g_i - \frac{1}{pq}\right) = 0 \tag{5}$$

Hence we introduce the two-parameter Hecke algebra $H_{p,q,n}$ generated by

$$g_i = P^{-1}R_i^{-1}, \qquad i = 1, 2, \ldots, n - 1$$

and the associated relations

$$g_i g_j = g_j g_i, \qquad |i - j| \geq 2 \tag{6}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{7}$$

in addition to relation (5), which can be rewritten in the form

$$g_i^2 = \left(\frac{1}{pq} - 1\right) g_i + \frac{1}{pq} \tag{8}$$

But it is known that (Jones, 1986) if $g_k = ae_k + b$ are elements of a Hecke algebra $A_{\beta,n}$ for $k = 1, 2, \ldots, n$, then

$$g_k g_{k+1} g_k = g_{k+1} g_k g_{k+1} \Leftrightarrow \beta^{-1} a^2 + ab + b^2 = 0$$

Also recalling the fact (Jones, 1986) that if e, f are elements of an algebra with identity 1, and if β, k are scalars satisfying $\beta = 2 + k + k^{-1}$, then for $g = (k+1)e - 1$, $h = (k+1)f - 1$, we have $e^2 = e \Leftrightarrow g^2 = (k-1)g + k$ and if $e^2 = e$, $f^2 = f$, then

$$efe = \beta^{-1} e \Leftrightarrow ghg + gh + hg + h + g + 1 = 0$$

Hence let us take $e_i = (g_i + 1)(pq/1 + pq)$, so $g_1 = (1 + 1/pq)e_i - 1$. Then for $a = (1 + 1/pq)$, $b = -1$, we find that (7), (8) are satisfied if and only if

$$\beta = 2 + \frac{1}{pq} + pq = (1 + pq)(1 + p^{-1}q^{-1})$$

Now consider the whole Hecke algebra $H_\infty = \cup_n H_{\beta,n}$; then, according to Ocneanu (see Jones, 1986), for any complex number $z \in C$ there is a trace function, Tr: $H_\infty(\beta) \to C$, uniquely defined by linearity and

$$\mathrm{Tr}(I) = 1 \tag{9}$$

$$\mathrm{tr}(ab) = \mathrm{tr}(ba) \tag{10}$$

$$\mathrm{tr}(wg_{n+1}) = z\,\mathrm{tr}(w), \qquad w \in H_{\beta,n} \tag{11}$$

So it is very convenient to introduce a polynomial invariant for knots and links. From (8), we write

$$g_i^{-1} = pqg_i + (pq - 1) \tag{12}$$

Hence from the braid relation (1) and the trace condition (10), a way to obtain the invariant is to normalize the g_i and g_i^{-1} so that Markov moves affect the trace in the same way, so that

$$\mathrm{tr}(kg_i) = \mathrm{tr}(kg_i)^{-1} = \frac{1}{k}\mathrm{tr}(g_i)^{-1}$$

Then,

$$kz = \frac{1}{k}[pqz + pq - 1], \qquad k^2 = \frac{pq(z+1) - 1}{z} \tag{13}$$

Thus

$$\mathrm{Tr}\left(\left\{\frac{pq(z+1) - 1}{z}\right\}^{1/2} g_i\right) = \mathrm{tr}\left(\left\{\frac{pq(z+1) - 1}{z}\right\}^{1/2} g_i\right)^{-1}$$

and

$$\mathrm{tr}(kg_i) = zk = z\left\{\frac{pq(z+1)-1}{z}\right\}^{1/2} = \{z[pq(z+1)-1]\}^{1/2}$$

Then, if we represent B_n by π_k: $B_n \to H(p, q, n)$ such that $\pi_k(\sigma_i) = kg_i$, then for every $\alpha \in B_n$, we have a function, $(kz)^{1-n}\,\mathrm{tr}(\pi(\alpha))$, which depends only on the link type α. Hence we can state the following result.

Theorem. For a link (knot) L with a braid representative $\alpha \in B_n$, the three-variable invariant

$$G_L(p, q, z) = \{z[pq(z+1)-1]\}^{(e-n+1)/2}\, z^{-e}\,\mathrm{tr}(\pi(\alpha))$$

where e is the exponent sum of α as a word on the generators of B_n, and $\pi(\sigma_I) = g_i$.

Example 1. For the (right-hand) trefoil T,

$$\begin{aligned}
G_T(p, q, z) &= \{z[pq(z+1)-1\}\, z^{-3}\,\mathrm{tr}(g_1^3) \\
&= z^{-2}[pq(z+1)-1]\,\mathrm{tr}\left\{\left(\frac{1}{pq}-1\right)g_1^2 + \frac{1}{pq}g_1\right\} \\
&= z^{-2}[pq(z+1)-1]\left\{\left[\left(\frac{1}{pq}-1\right)^2 + \frac{1}{pq}\right]z + \frac{1}{pq}\left(\frac{1}{pq}-1\right)\right\} \\
&= \frac{1}{p^2q^2z^2}[pq(z+1)-1][(1-pq+p^2q^2)z + (1-pq)]
\end{aligned}$$

Example 2. The figure-eight knot L is given by the closure of the braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \in B_3$. Then,

$$\begin{aligned}
&G_L(p, q, z) \\
&\quad = \{z[pq(z+1)-1]\}^{-1}\,\mathrm{tr}(g_1g_2^{-1}g_1g_2^{-1}) \\
&\quad = \{z[pq(z+1)-1]\}^{-1}\,\mathrm{tr}\{p^2q^2g_1g_2g_1g_2 + pq(pq-1)g_1^2g_2 \\
&\qquad + pq(pq-1)g_1g_2g_1 + (pq-1)^2g_1^2\} \\
&\quad = \{z[pq(z+1)-1]\}^{-1}p^2q^2\,\mathrm{tr}\left\{g_1^3g_2 + 2\left(1-\frac{1}{pq}\right)g_1^2g_2\left(1-\frac{1}{pq}\right)^2 g_1^2\right\} \\
&\quad = \{z[pq(z+1)-1]\}^{-1}p^2q^2\,\mathrm{tr}\left\{zg_1^3 + \left[2\left(1-\frac{1}{pq}\right)z + \left(1-\frac{1}{pq}\right)^2\right]\mathrm{tr}\,g_1^2\right\} \\
&\quad = \{z[pq(z+1)-1]\}^{-1}p^2q^2\{z(pq)^2\{(z+1)(1-pq) + zp^2q^2\}
\end{aligned}$$

$$
+\left[2\left(1-\frac{1}{pq}\right)z+\left(1-\frac{1}{pq}\right)^2\right](pq)^{-1}[(z+1)+z(-pq)]\Bigg\}
$$

$$
=\frac{1}{pqz[pq(z+1)-1]}\{[p^2q^2-2pq+1]-z[p^3q^3-4p^2q^2+4pq]
$$

$$
-z^2[p^3q^3-3p^2q^2+pq]\}
$$

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